Diploma Programme
Programme du diplôme
Programa del Diploma

# Markscheme

## November 2021

## Mathematics: analysis and approaches

## Higher level

## Paper 2

32 pages

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## Instructions to Examiners

### Abbreviations

***M*** Marks awarded for attempting to use a correct **Method**.

***A*** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

***R*** Marks awarded for clear **Reasoning**.

***AG*** Answer given in the question and so no marks are awarded.

***FT*** Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

### Using the markscheme

#### 1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2.***

#### 2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a “show that” question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

| | **Correct answer seen** | **Further working seen** | **Any FT issues?** | **Action** |
|---|---|---|---|---|
| **1.** | $8\sqrt{2}$ | $5.65685...$ *(incorrect decimal value)* | No. Last part in question. | Award ***A1*** for the final mark *(condone the incorrect further working)* |
| **2.** | $\frac{35}{72}$ | $0.468111\ldots$ (incorrect decimal value) | Yes. Value is used in subsequent parts. | Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)* |

**3 Implied marks**

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

**4 Follow through marks (only applied after an error is made)**

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

**For example**: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This

includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

## 5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than $1$, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

## 6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

## 7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example $1.9$ and $1,9$ or $1000$ and $1,000$ and $1.000$.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

## 8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

**Simplification of final answers:** Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as $2$, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

**Please note:** intermediate ***A*** marks do NOT need to be simplified.

**9 Calculators**

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

**10. Presentation of candidate work**

**Crossed out work:** If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

**More than one solution:** Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

## Section A

**1.** (a) use of GDC to give ***(M1)***

$r = 0.883529...$

$r = 0.884$ ***A1***

| |
|---|
| **Note:** Award the ***(M1)*** for any correct value of $r, a, b$ or $r^2 = 0.780624...$ seen in part (a) or part (b). |

***[2 marks]***

(b) $a = 1.36609...$ , $b = 64.5171...$

$a = 1.37$ , $b = 64.5$ ***A1***

***[1 mark]***

(c) attempt to find their difference ***(M1)***

$5 \times 1.36609...$ OR $1.36609...(h+5) + 64.5171... - (1.36609...h + 64.5171...)$

$6.83045...$

$= 6.83 (6.85 \text{ from } 1.37)$

the student could have expected her score to increase by 7 marks. ***A1***

| |
|---|
| **Note:** Accept an increase of 6, 6.83 or 6.85. |

***[2 marks]***

(d) Lucy is incorrect in suggesting there is a causal relationship.
This might be true, but the data can only indicate a correlation. ***R1***

| |
|---|
| **Note:** Accept 'Lucy is incorrect as correlation does not imply causation' or equivalent. |

***[1 mark]***

(e) no effect ***A1***

***[1 mark]***

***Total [7 marks]***

**2.** **EITHER**

attempt to use cosine rule ***(M1)***

$12^2 + AB^2 - 2\times 12\times \cos 25° \times AB = 7^2$ OR $AB^2 - 21.7513...AB + 95 = 0$ ***(A1)***

at least one correct value for $AB$ ***(A1)***

$AB = 6.05068...$ OR $AB = 15.7007...$

using their smaller value for $AB$ to find minimum perimeter ***(M1)***

$12 + 7 + 6.05068...$

**OR**

attempt to use sine rule ***(M1)***

$\frac{\sin B}{12} = \frac{\sin 25°}{7}$ OR $\sin B = 0.724488...$ OR $\hat{B} = 133.573...°$ OR $\hat{B} = 46.4263...°$ ***(A1)***

at least one correct value for $\hat{C}$

$\hat{C} = 21.4263...°$ OR $\hat{C} = 108.573...°$ ***(A1)***

using their acute value for $\hat{C}$ to find minimum perimeter ***(M1)***

$12 + 7 + \sqrt{12^2 + 7^2 - 2\times 12\times 7\cos 21.4263...°}$ OR $12 + 7 + \frac{7\sin 21.4263...°}{\sin 25°}$

**THEN**

$25.0506\ldots$

minimum perimeter $= 25.1$. ***A1***

***Total [5 marks]***

**3.** (a) recognize that the variable has a Binomial distribution ***(M1)***

$X \sim \mathrm{B}(30, 0.05)$

attempt to find $\mathrm{P}(X \geq 1)$ ***(M1)***

$1 - \mathrm{P}(X = 0)$ OR $1 - 0.95^{30}$ OR $1 - 0.214638...$ OR $0.785361...$

| **Note:** The two ***M*** marks are independent of each other. |
|---|

$\mathrm{P}(X \geq 1) = 0.785$ ***A1***

***[3 marks]***

(b) recognition of conditional probability ***(M1)***

$\mathrm{P}(X \leq 2 \mid X \geq 1)$ OR $\mathrm{P}(\text{at most 2 defective} \mid \text{at least 1 defective})$

| **Note:** Recognition must be shown in context either in words or symbols but not just $\mathrm{P}(A \mid B)$. |
|---|

$\dfrac{\mathrm{P}(1 \leq X \leq 2)}{\mathrm{P}(X \geq 1)}$ OR $\dfrac{\mathrm{P}(X = 1) + \mathrm{P}(X = 2)}{\mathrm{P}(X \geq 1)}$ ***(A1)***

$\dfrac{0.597540...}{0.785361...}$ OR $\dfrac{0.812178... - 0.214638...}{0.785361...}$ OR $\dfrac{0.338903... + 0.258636...}{0.785361...}$ ***(A1)***

$= 0.760847...$

$\mathrm{P}(X \leq 2 \mid X \geq 1) = 0.761$ ***A1***

***[4 marks]***

***Total [7 marks]***

**4.** (a) attempt to find the area of either shaded region in terms of $r$ and $\theta$ ***(M1)***

| **Note*:*** Do not award ***M1*** if they have only copied from the booklet and not applied to the shaded area. |
|---|

Area of segment $=\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta$ ***A1***

Area of triangle $=\frac{1}{2}r^2\sin(\pi-\theta)$ ***A1***

correct equation in terms of $\theta$ only ***(A1)***

$\theta-\sin\theta=\sin(\pi-\theta)$

$\theta-\sin\theta=\sin\theta$ ***A1***

$\theta=2\sin\theta$ ***AG***

| **Note*:*** Award a maximum of ***M1A1A0A0A0*** if a candidate uses degrees (i.e., $\frac{1}{2}r^2\sin(180°-\theta)$), even if later work is correct. |
|---|

| **Note*:*** If a candidate directly states that the area of the triangle is $\frac{1}{2}r^2\sin\theta$, award a maximum of ***M1A1A0A1A1***. |
|---|

***[5 marks]***

(b) $\theta=1.89549...$

$\theta=1.90$ ***A1***

| **Note:** Award ***A0*** if there is more than one solution. Award ***A0*** for an answer in degrees. |
|---|

***[1 mark]***

***Total [6 marks]***

**5.** (a) $u_1 = S_1 = \frac{2}{3} \times \frac{7}{8}$ ***(M1)***

$= \frac{14}{24}\left(= \frac{7}{12} = 0.583333\ldots\right)$ ***A1***

***[2 marks]***

(b) $r = \frac{7}{8}(= 0.875)$ ***(A1)***

substituting their values for $u_1$ and $r$ into $S_\infty = \frac{u_1}{1-r}$ ***(M1)***

$= \frac{14}{3}(= 4.66666\ldots)$ ***A1***

***[3 marks]***

(c) attempt to substitute their values into the inequality or formula for $S_n$ ***(M1)***

$\frac{14}{3} - \sum_{r=1}^{n} \frac{2}{3}\left(\frac{7}{8}\right)^r < 0.001$ OR $S_n = \frac{\frac{7}{12}\left(1-\left(\frac{7}{8}\right)^n\right)}{\left(1-\frac{7}{8}\right)}$

attempt to solve their inequality using a table, graph or logarithms
(must be exponential) ***(M1)***

| |
|---|
| **Note:** Award ***(M0)*** if the candidate attempts to solve $S_\infty - u_n < 0.001$. |

correct critical value or at least one correct crossover value ***(A1)***

$63.2675\ldots$ OR $S_\infty - S_{63} = 0.001036...$ OR $S_\infty - S_{64} = 0.000906...$

OR $S_\infty - S_{63} - 0.001 = 0.0000363683...$ OR $S_\infty - S_{64} - 0.001 = -0.0000931777...$

least value is $n = 64$ ***A1***

***[4 marks]***

***Total [9 marks]***

**6.** (a) **METHOD 1**

$(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$

attempts to expand $(p+q)^3$ ***M1***

$p^3 + 3p^2q + 3pq^2 + q^3$

$(p+q)^3 - 3pq(p+q) \equiv p^3 + 3p^2q + 3pq^2 + q^3 - 3pq(p+q)$

$\equiv p^3 + 3p^2q + 3pq^2 + q^3 - 3p^2q - 3pq^2$ ***A1***

$\equiv p^3 + q^3$ ***AG***

**Note*:*** Condone the use of equals signs throughout.

**METHOD 2**

$(p+q)^3 - 3pq(p+q) \equiv p^3 + q^3$

attempts to factorise $(p+q)^3 - 3pq(p+q)$ ***M1***

$\equiv (p+q)\left((p+q)^2 - 3pq\right) \quad \left(\equiv (p+q)\left(p^2 - pq + q^2\right)\right)$

$\equiv p^3 - p^2q + pq^2 + p^2q - pq^2 + q^3$ ***A1***

$\equiv p^3 + q^3$ ***AG***

**Note*:*** Condone the use of equals signs throughout.

**METHOD 3**

$p^3 + q^3 \equiv (p+q)^3 - 3pq(p+q)$

attempts to factorise $p^3 + q^3$ ***M1***

$\equiv (p+q)\left(p^2 - pq + q^2\right)$

$\equiv (p+q)\left((p+q)^2 - 3pq\right)$ ***A1***

$\equiv (p+q)^3 - 3pq(p+q)$ ***AG***

**Note*:*** Condone the use of the equals sign throughout.

***[2 marks]***

(b)

> **Note:** Award a maximum of ***A1M0A0A1M0A0*** for $m = -95$ and $n = 8$ found by using $\alpha, \beta = \frac{5 \pm \sqrt{17}}{4}$ $(\alpha, \beta = 0.219..., 2.28...)$.
> Condone, as appropriate, solutions that state but clearly do not use the values of $\alpha$ and $\beta$.
> Special case: Award a maximum of ***A1M1A0A1M0A0*** for $m = -95$ and $n = 8$ obtained by solving simultaneously for $\alpha$ and $\beta$ from product of roots and sum of roots equations.

product of roots of $x^2 - \frac{5}{2}x + \frac{1}{2} = 0$

$\alpha\beta = \frac{1}{2}$ (seen anywhere) ***A1***

considers $\left(\frac{1}{\alpha^3}\right)\left(\frac{1}{\beta^3}\right)$ by stating $\frac{1}{(\alpha\beta)^3} (= n)$ ***M1***

> **Note:** Award ***M1*** for attempting to substitute their value of $\alpha\beta$ into $\frac{1}{(\alpha\beta)^3}$.

$$\frac{1}{(\alpha\beta)^3} = \frac{1}{\left(\frac{1}{2}\right)^3}$$

$n = 8$ ***A1***

sum of roots of $x^2 - \frac{5}{2}x + \frac{1}{2} = 0$

$\alpha + \beta = \frac{5}{2}$ (seen anywhere) ***A1***

considers $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$ by stating $\frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$ $\left(\left(\frac{\alpha+\beta}{\alpha\beta}\right)^3 - \frac{3(\alpha+\beta)}{(\alpha\beta)^2}\right) (= -m)$ ***M1***

> **Note:** Award ***M1*** for attempting to substitute their values of $\alpha + \beta$ and $\alpha\beta$ into their expression. Award ***M0*** for use of $(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$ only.

$$= \frac{\left(\frac{5}{2}\right)^3 - \left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}{\frac{1}{8}} \quad (= 125 - 30 = 95)$$

$m = -95$ ***A1***

$(x^2 - 95x + 8 = 0)$

***[6 marks]***
***Total [8 marks]***

**7.** (a) recognises that $\int_0^m \arccos x \, dx = 0.5$ ***(M1)***

$m \arccos m - \sqrt{1-m^2} - \left(0 - \sqrt{1}\right) = 0.5$

$m = 0.360034...$

$m = 0.360$ ***A1***

***[2 marks]***

(b) **METHOD 1**

attempts to find at least one endpoint (limit) both in terms of $m$ (or their $m$) and $a$ ***(M1)***

$\mathrm{P}\left(m - a \le X \le m + a\right) = 0.3$

$\int_{0.360034...-a}^{0.360034...+a} \arccos x \, dx = 0.3$ ***(A1)***

| |
|---|
| **Note***:* Award ***(A1)*** for $\int_{m-a}^{m+a} \arccos x \, dx = 0.3$. |

$\left[x \arccos x - \sqrt{1-x^2}\right]_{0.360034...-a}^{0.360034...+a}$

attempts to solve their equation for $a$ ***(M1)***

| |
|---|
| **Note***:* The above ***(M1)*** is dependent on the first ***(M1)***. |

$a = 0.124861...$

$a = 0.125$ ***A1***

**METHOD 2**

$\int_{-a}^{a} \arccos\left|x - 0.360034...\right| \, dx \ (= 0.3)$ ***(M1)(A1)***

| |
|---|
| **Note:** Only award ***(M1)*** if at least one limit has been translated correctly. |

| |
|---|
| **Note:** Award ***(M1)(A1)*** for $\int_{-a}^{a} \arccos\left|x - m\right| \, dx \ (= 0.3)$. |

attempts to solve their equation for $a$ ***(M1)***

$a = 0.124861...$

$a = 0.125$ ***A1***

**METHOD 3**

**EITHER**

$$\int_{-a}^{a} \arccos\left(x+0.360034...\right)\,\mathrm{d}x \ \left(=0.3\right)$$ ***(M1)(A1)***

| |
|---|
| **Note:** Only award ***(M1)*** if at least one limit has been translated correctly. |

| |
|---|
| **Note:** Award ***(M1)(A1)*** for $\int_{-a}^{a} \arccos\left(x+m\right)\,\mathrm{d}x \ \left(=0.3\right)$. |

**OR**

$$\int_{2(0.360034...)-a}^{2(0.360034...)+a} \arccos\left(x-0.360034...\right)\,\mathrm{d}x \ \left(=0.3\right)$$ ***(M1)(A1)***

| |
|---|
| **Note:** Only award ***(M1)*** if at least one limit has been translated correctly. |

| |
|---|
| **Note:** Award ***(M1)(A1)*** for $\int_{2m-a}^{2m+a} \arccos\left(x-m\right)\,\mathrm{d}x \ \left(=0.3\right)$. |

**THEN**

attempts to solve their equation for $a$ ***(M1)***

| |
|---|
| **Note*:*** The above ***(M1)*** is dependent on the first ***(M1)***. |

$a=0.124861...$

$a=0.125$ ***A1***

***[4 marks]***

***Total [6 marks]***

**8.** (a) **METHOD 1**

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$u = xy$ , $v = \ln(xy)$, $\frac{\mathrm{d}u}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$ , $\frac{\mathrm{d}v}{\mathrm{d}x} = \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\frac{1}{xy}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left[\frac{xy}{xy}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln(xy)\right]$ ***A1***

**Note*:*** Award ***(M1)A1*** for implicitly differentiating $y = x(1 - y\ln(xy))$ and obtaining

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left[\frac{xy}{xy}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + x\frac{\mathrm{d}y}{\mathrm{d}x}\ln(xy) + y\ln(xy)\right]$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left[\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right) + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln(xy)\right]$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy))$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy)) = 1$ ***AG***

**METHOD 2**

$y = x - xy\ln x - xy\ln y$

attempts to differentiate implicitly including at least one application of the product rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(\frac{xy}{x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln x\right) - \left(\frac{xy}{y}\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln y\right)$ ***A1***

or equivalent to the above, for example

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(x\ln x\frac{\mathrm{d}y}{\mathrm{d}x} + (1 + \ln x)y\right) - \left(y\ln y + x\left(\ln y\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x}\right)\right)$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x\frac{\mathrm{d}y}{\mathrm{d}x}(\ln x + \ln y + 1) - y(\ln x + \ln y + 1)$ ***A1***

or equivalent to the above, for example

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x\frac{\mathrm{d}y}{\mathrm{d}x}(\ln(xy) + 1) - y(\ln(xy) + 1)$

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)(1 + \ln(xy)) = 1$ ***AG***

**METHOD 3**

attempt to differentiate implicitly including at least one application of the product rule ***M1***

$u = x\ln(xy),\ v = y,\ \frac{\mathrm{d}u}{\mathrm{d}x} = \ln(xy) + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\frac{x}{xy},\ \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x}\ln(xy) + y\ln(xy) + \frac{xy}{xy}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\right)$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - x\frac{\mathrm{d}y}{\mathrm{d}x}\left(\ln(xy) + 1\right) - y\left(\ln(xy) + 1\right)$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\left(1 + \ln(xy)\right) = 1$ ***AG***

**METHOD 4**

lets $w = xy$ and attempts to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $y = x - w\ln w$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}w}{\mathrm{d}x}\ln w\right)\ \left(= 1 - \frac{\mathrm{d}w}{\mathrm{d}x}(1 + \ln w)\right)$ ***A1***

$\frac{\mathrm{d}w}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\ln(xy)\right)\ \left(= 1 - \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\left(1 + \ln(xy)\right)\right)$

$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right)\left(1 + \ln(xy)\right) = 1$ ***AG***

***[3 marks]***

(b) **METHOD 1**

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$ ***A1***

substitutes $x=1$ and their non-zero value of $y$ into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(xy))=1$ ***(M1)***

$2\frac{\mathrm{d}y}{\mathrm{d}x}=0 \ \left(\frac{\mathrm{d}y}{\mathrm{d}x}=0\right)$ ***A1***

equation of the tangent is $y=1$ ***A1***

**METHOD 2**

substitutes $x=1$ into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(xy))=1$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(y))=1$

**EITHER**

correctly substitutes $\ln y=\frac{1-y}{y}$ into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(y))=1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+\frac{1}{y}\right)=0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=0 \ (y=1)$ ***A1***

**OR**

correctly substitutes $y+y\ln y=1$ into $\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)(1+\ln(y))=1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x}(2+\ln y)=0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=0 \ (y=1)$ ***A1***

**THEN**

substitutes $x=1$ into $y=x-xy\ln(xy)$ ***(M1)***

$y=1-y\ln y \Rightarrow y=1$

equation of the tangent is $y=1$ ***A1***

***[5 marks]***

***Total [8 marks]***

## Section B

**9.** (a) $12 = \dfrac{2\pi}{b}$ OR $b = \dfrac{2\pi}{12}$ ***A1***

$b = \dfrac{\pi}{6}$ ***AG***

***[1 mark]***

(b) $a = \dfrac{6.8 - 2.2}{2}$ OR $a = \dfrac{\max - \min}{2}$ ***(M1)***

$= 2.3$ (m) ***A1***

***[2 marks]***

(c) $d = \dfrac{6.8 + 2.2}{2}$ OR $d = \dfrac{\max + \min}{2}$ ***(M1)***

$= 4.5$ (m) ***A1***

***[2 marks]***

*continued…*

*Question 9 continued.*

(d) **METHOD 1**

substituting $t = 4.5$ and $H = 6.8$ for example into their equation for $H$ ***(A1)***

$$6.8 = 2.3\sin\left(\frac{\pi}{6}(4.5 - c)\right) + 4.5$$

attempt to solve their equation ***(M1)***

$c = 1.5$ ***A1***

**METHOD 2**

using horizontal translation of $\frac{12}{4}$ ***(M1)***

$4.5 - c = 3$ ***(A1)***

$c = 1.5$ ***A1***

**METHOD 3**

$H'(t) = (2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(t - c)\right)$ ***(A1)***

attempts to solve their $H'(4.5) = 0$ for $c$ ***(M1)***

$$(2.3)\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}(4.5 - c)\right) = 0$$

$c = 1.5$ ***A1***

***[3 marks]***

(e) attempt to find $H$ when $t = 12$ or $t = 0$, graphically or algebraically ***(M1)***

$H = 2.87365...$

$H = 2.87\,(\text{m})$ ***A1***

***[2 marks]***

*continue…*

*Question 9 continued.*

(f) attempt to solve $5 = 2.3\sin\left(\frac{\pi}{6}(t-1.5)\right)+4.5$ ***(M1)***

times are $t = 1.91852...$ and $t = 7.08147...$, $(t = 13.9185..., t = 19.0814...)$ ***(A1)***

total time is $2\times(7.081...-1.919...)$

$10.3258...$

$=10.3$ (hours) ***A1***

| **Note:** Accept 10. |
|---|

***[3 marks]***

(g) **METHOD 1**

substitutes $t = \frac{11}{3}$ and $H = 6.8$ into their equation for $H$ and attempts to solve for $c$ ***(M1)***

$6.8 = 2.3\sin\left(\frac{\pi}{6}\left(\frac{11}{3}-c\right)\right)+4.5 \Rightarrow c = \frac{2}{3}$

$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t-\frac{2}{3}\right)\right)+4.5$ ***A1***

**METHOD 2**

uses their horizontal translation $\left(\frac{12}{4}=3\right)$ ***(M1)***

$\frac{11}{3}-c = 3 \Rightarrow c = \frac{2}{3}$

$H(t) = 2.3\sin\left(\frac{\pi}{6}\left(t-\frac{2}{3}\right)\right)+4.5$ ***A1***

***[2 marks]***
***Total [15 marks]***

**10.** (a) (i)

> **Note:** In part (a), penalise once only, if correct values are given instead of correct coordinates.

attempts to solve $x^2 - x - 12 = 0$ ***(M1)***

$(-3, 0)$ and $(4, 0)$ ***A1***

(ii) $\left(0, \frac{4}{5}\right)$ ***A1***

***[3 marks]***

(b) $x = \frac{15}{2}$ ***A1***

> **Note:** Award ***A0*** for $x \neq \frac{15}{2}$.
>
> Award ***A1*** in part (b), if $x = \frac{15}{2}$ is seen on their graph in part (d).

***[1 mark]***

(c) **METHOD 1**

$(ax + b)(2x - 15) \equiv x^2 - x - 12$

attempts to expand $(ax + b)(2x - 15)$ ***(M1)***

$2ax^2 - 15ax + 2bx - 15b \equiv x^2 - x - 12$

$a = \frac{1}{2}$ ***A1***

equates coefficients of $x$ ***(M1)***

$-1 = -\frac{15}{2} + 2b$

$b = \frac{13}{4}$ ***A1***

$\left(y = \frac{x}{2} + \frac{13}{4}\right)$

**METHOD 2**

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$\frac{x}{2}+\frac{13}{4}+...$ ***M1***

$a=\frac{1}{2}$ ***A1***

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

**METHOD 3**

$a=\frac{1}{2}$ ***A1***

$\frac{x^2-x-12}{2x-15}\equiv\frac{x}{2}+b+\frac{c}{2x-15}$ ***M1***

$x^2-x-12\equiv\frac{(2x-15)x}{2}+(2x-15)b+c$

equates coefficients of $x$ : ***(M1)***

$-1=-\frac{15}{2}+2b$

$b=\frac{13}{4}$ ***A1***

$\left(y=\frac{x}{2}+\frac{13}{4}\right)$

**METHOD 4**

attempts division on $\frac{x^2-x-12}{2x-15}$ ***M1***

$$\frac{x^2-x-12}{2x-15}=\frac{x}{2}+\frac{\frac{13x}{2}-12}{2x-15}$$

$a=\frac{1}{2}$ ***A1***

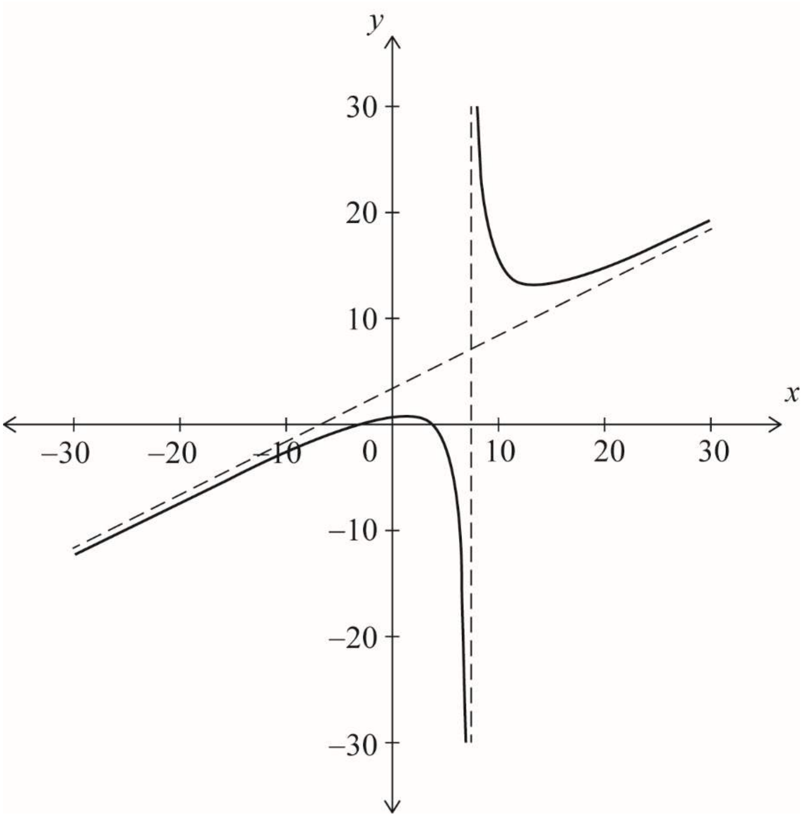
$$\frac{\frac{13x}{2}-12}{2x-15}=\frac{13}{4}+\ldots$$ ***M1***

$b=\frac{13}{4}$ ***A1***

$$\left(y=\frac{x}{2}+\frac{13}{4}\right)$$

***[4 marks]***

(d)




two branches with approximately correct shape ( for $-30 \le x \le 30$ ) ***A1***
their vertical and oblique asymptotes in approximately correct positions with both branches showing correct asymptotic behaviour to these asymptotes ***A1***
their axes intercepts in approximately the correct positions ***A1***

**Note:** Points of intersection with the axes and the equations of asymptotes are not required to be labelled.

***[3 marks]***

(e) (i) attempts to split into partial fractions: ***(M1)***

$$\frac{2x-15}{(x+3)(x-4)} \equiv \frac{A}{x+3} + \frac{B}{x-4}$$

$$2x-15 \equiv A(x-4) + B(x+3)$$

$A = 3$ ***A1***

$B = -1$ ***A1***

$$\left(\frac{3}{x+3} - \frac{1}{x-4}\right)$$

(ii) $\int_0^3 \left(\frac{3}{x+3} - \frac{1}{x-4}\right) \mathrm{d}x$

attempts to integrate and obtains two terms involving 'ln' ***(M1)***

$= \left[3\ln|x+3| - \ln|x-4|\right]_0^3$ ***A1***

$= 3\ln 6 - \ln 1 - 3\ln 3 + \ln 4$ ***A1***

$= 3\ln 2 + \ln 4 \ (= \ln 8 + \ln 4)$

$= \ln 32 \ (= 5\ln 2)$ ***A1***

| |
|---|
| **Note:** The final ***A1*** is dependent on the previous two ***A*** marks. |

***[7 marks]***

***Total [18 marks]***

**11.** (a) (i) attempts to find either $\vec{AB}$ or $\vec{AC}$ ***(M1)***

$\vec{AB} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$ and $\vec{AC} = \begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ ***A1***

(ii) **METHOD 1**

attempts to find $\vec{AB} \times \vec{AC}$ ***(M1)***

$\vec{AB} \times \vec{AC} = \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix}$ ***A1***

**EITHER**

equation of plane is of the form $14x - 21y - 7z = d$ $(2x - 3y - z = d)$ ***(A1)***

substitutes a valid point e.g $(3,0,0)$ to obtain a value of $d$ ***M1***

$d = 42$ $(d = 6)$

**OR**

attempts to use $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ ***(M1)***

$\boldsymbol{r} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix}$ $\left(\boldsymbol{r} \cdot \begin{pmatrix} 14 \\ -21 \\ -7 \end{pmatrix} = 42\right)$ ***A1***

$\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$ $\left(\boldsymbol{r} \cdot \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 6\right)$

**THEN**

$14x - 21y - 7z = 42$ $(2x - 3y - z = 6)$ ***A1***

**METHOD 2**

equation of plane is of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -7 \end{pmatrix}$ ***A1***

attempts to form equations for $x, y, z$ in terms of their parameters ***(M1)***

$x = 3 - 3s - 2t$, $y = -2s + t$, $z = -7t$

***A1***

eliminates at least one of their parameters ***(M1)***

for example, $2x - 3y = 6 - 7t$ $(\Rightarrow 2x - 3y = 6 + z)$

$2x - 3y - z = 6$ ***A1***

***[7 marks]***

(b) **METHOD 1**

substitutes $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ into their $\Pi_1$ and $\Pi_2$ (given) ***(M1)***

$\Pi_1$: $2\lambda - 3(-2+\lambda) - (-\lambda) = 6$ and $\Pi_2$: $3\lambda - (-2+\lambda) + 2(-\lambda) = 2$ ***A1***

**Note:** Award ***(M1)A0*** for correct verification using a specific value of $\lambda$.

so the vector equation of $L$ can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ ***AG***

**METHOD 2**
**EITHER**

attempts to find $\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ ***M1***

$= \begin{pmatrix} -7 \\ -7 \\ 7 \end{pmatrix}$

**OR**

$\begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (2-3+1) = 0$ and $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = (3-1-2) = 0$ ***M1***

**THEN**

substitutes $(0, -2, 0)$ into $\Pi_1$ and $\Pi_2$

$\Pi_1$: $2(0) - 3(-2) - (0) = 6$ and $\Pi_2$: $3(0) - (-2) + 2(0) = 2$ ***A1***

so the vector equation of $L$ can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ ***AG***

**METHOD 3**

attempts to solve $2x - 3y - z = 6$ and $3x - y + 2z = 2$ ***(M1)***

for example, $x = -\lambda, y = -2 - \lambda, z = \lambda$ ***A1***

| |
|---|
| Note: Award ***A1*** for substituting $x=0$ (or $y=-2$ or $z=0$) into $\Pi_1$ and $\Pi_2$ and solving simultaneously. For example, solving $-3y-z=6$ and $-y+2z=2$ to obtain $y=-2$ and $z=0$. |

so the vector equation of $L$ can be written as $\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$ ***AG***

***[2 marks]***

(c) (i) substitutes the equation of $L$ into the equation of $\Pi_3$ ***(M1)***

$2\lambda+2\lambda=3\Rightarrow 4\lambda=3$ ***A1***

$\lambda=\dfrac{3}{4}$ ***AG***

(ii) $\mathrm{P}$ has coordinates $\left(\dfrac{3}{4},-\dfrac{5}{4},-\dfrac{3}{4}\right)$ ***A1***

***[3 marks]***

(d) (i) normal to $\Pi_3$ is $\boldsymbol{n}=\begin{pmatrix}2\\0\\-2\end{pmatrix}$ ***(A1)***

| |
|---|
| **Note:** May be seen or used anywhere. |

considers the line normal to $\Pi_3$ passing through $\mathrm{B}(0,-2,0)$ ***(M1)***

$\boldsymbol{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\mu\begin{pmatrix}2\\0\\-2\end{pmatrix}$ ***A1***

**EITHER**

finding the point on the normal line that intersects $\Pi_3$

attempts to solve simultaneously with plane $2x-2z=3$ ***(M1)***

$4\mu+4\mu=3$

$\mu=\dfrac{3}{8}$ ***A1***

point is $\left(\dfrac{3}{4},-2,-\dfrac{3}{4}\right)$

**OR**

$$\left(\begin{pmatrix} 2\mu \\ -2 \\ -2\mu \end{pmatrix} - \begin{pmatrix} \frac{3}{4} \\ -\frac{5}{4} \\ -\frac{3}{4} \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = 0$$ ***(M1)***

$$4\mu - \frac{3}{2} + 4\mu - \frac{3}{2} = 0$$

$$\mu = \frac{3}{8}$$ ***A1***

**OR**

attempts to find the equation of the plane parallel to $\Pi_3$ containing $\mathrm{B}'$ $(x - z = 3)$ and solve simultaneously with $L$ ***(M1)***

$$2\mu' + 2\mu' = 3$$

$$\mu' = \frac{3}{4}$$ ***A1***

**THEN**

so, another point on the reflected line is given by

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \frac{3}{4}\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$ ***(A1)***

$$\Rightarrow \mathrm{B}'\left(\frac{3}{2}, -2, -\frac{3}{2}\right)$$ ***A1***

(ii) **EITHER**

attempts to find the direction vector of the reflected line using their P and B′ ***(M1)***

$$\vec{\mathrm{PB}'} = \begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$

**OR**

attempts to find their direction vector of the reflected line using a vector approach ***(M1)***

$$\vec{\mathrm{PB}'} = \vec{\mathrm{PB}} + \vec{\mathrm{BB}'} = -\frac{3}{4}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

**THEN**

$$\boldsymbol{r} = \begin{pmatrix} \frac{3}{2} \\ -2 \\ -\frac{3}{2} \end{pmatrix} + \lambda\begin{pmatrix} \frac{3}{4} \\ -\frac{3}{4} \\ -\frac{3}{4} \end{pmatrix}$$ (or equivalent) ***A1***

**Note:** Award ***A0*** for either '$\boldsymbol{r} =$' or '$\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$' not stated. Award ***A0*** for '$L' =$'.

***[9 marks]***
***Total [21 marks]***